JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 63, 446–462 (1978)

# On a Family of Approximation Operators*

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Starting from a defining differential equation $(\partial/\partial t)\, W(\lambda, t, u) = (\lambda(u-t)/p(t))\, W(\lambda, t, u)$ of the kernel of an exponential operator $S_\lambda(f, t) = \int_{-\infty}^{\infty} W(\lambda, t, u) f(u)\, du$ with normalization $\int_{-\infty}^{\infty} W(\lambda, t, u)\, du = 1$, we determine $S_\lambda$ for various $p(t)$ including; for example, $p(t)$ a quadratic polynomial, all the known exponential operators are recovered and some new ones are constructed. It is shown that all the exponential operators are approximation operators. Further approximation properties of these operators are discussed. For example, functions satisfying $\| S_\lambda(f, t) - f(t) \| = O(\lambda^{-\alpha})$ are characterized. Several results of C. P. May are also improved.

## 1. INTRODUCTION

Let $S_\lambda(f, t)$ be a positive operator defined by

$$S_\lambda(f, t) = \int_{-\infty}^{\infty} W(\lambda, t, u) f(u)\, du. \tag{1.1}$$

It is known that, in many real cases (see [7]), $S_\lambda$ satisfies the following homogeneous partial differential equation.

$$\frac{\partial}{\partial t} W(\lambda, t, u) = \frac{\lambda}{p(t)} W(\lambda, t, u)\,(u - t), \tag{1.2}$$

* This work was done at the University of Toronto and supported by Grant A4048 of the National Research Council of Canada.



0022-247X/78/0632–0446$02.00/0

where $p(t)$ is analytic and positive foı $t \in (A, B)$ for some $A, B, -\infty \leqslant A < B \leqslant +\infty$, and the normalization condition

$$S_\lambda(1, t) = \int_{-\infty}^{+\infty} W(\lambda, t, u)\, du = 1. \tag{1.3}$$

We shall always assume that the domain of $S_\lambda$ contains at least the polynomial functions. Note that the kernel $W(\lambda, t, u)$ may be a function or a generalized function and (1.2) means

$$\frac{\partial}{\partial t} \int_{-\infty}^{\infty} W(\lambda, t, u) f(u)\, du = \frac{\lambda}{p(t)} \int_{-\infty}^{\infty} W(\lambda, t, u)\,(u - t) f(u)\, du,$$

see Zemanian [13].

Operators satisfying (1.2) are, for example, the Bernstein polynomials and operators of Szász, Post–Widder, Gauss–Weierstrass and Baskakov (see, e.g., [1, 5, 11, 12]). These operators are also referred to as the exponential operators [7].

In [7], May has systematically studied some approximation properties when $p(t)$ is a polynomial of degree at most 2. The following question arises naturally: can we characterize all the exponential operators associated with a given $p(t)$? In particular, is there any more exponential operators other than the known ones when $p(t) = at^2 + bt + c$?

In this paper we prove that for each $p(t)$, the differential equation (1.2) and the normalization (1.3) determine uniquely the approximation operator $S_\lambda(f, t)$. As a consequence we show that the exponential operators associated with a constant, linear or quadratic $p(t)$ are the known operators discussed in [7] and the new operator defined by (3.10). Moreover, we give some new operators with cubic $p(t)$ as examples of solving (1.2). A more striking example is (3.16), where $p(t) = 2t^{3/2}$.

Some of the techniques used in [7] for a quadratic polynomial $p(t)$ can be applied to more general $p(t)$. We discuss briefly some of the approximation properties for these general cases. In fact, we show that all exponential operators are approximation operators. Functions which satisfy $\| S_\lambda(f, t) - f(t)\| = O(\lambda^{-\alpha/2})$, $0 < \alpha < 2$ for arbitrary exponential operator $S_\lambda$ are also characterized in this paper. The results in this part and the technique for showing it are similar to that of [7], but we achieve better "growth-test functions" than those shown in [7] (see Remark 2.6 later), which is an improvement.

It seems to us that the relations (1.2) and (1.3) are canonical forms for the associated approximation operator. Since only equivalent approximation processes can be derived from the oıiginal one by changing variables, we shall identify such operators. This identification may not be at all obvious from the explicit forms of the operators. For example, we show that the Meyer–König–Zeller operators [8] are indeed equivalent to the Baskakov operators, a fact that apparently has been overlooked.

## 2. Some Preliminary Properties

We prove some of the properties of the exponential operators in this section. These properties will be useful for further discussion.

Proposition 2.1. *Let $S_\lambda(\cdot, t)$ be an exponential operator. Then*

(i) $S_\lambda(1, t) = 1$;

(ii) $S_\lambda(u, t) = t$;

(iii) $S_\lambda(u^2, t) = t^2 + p(t)/\lambda$.

*Proof.* The relation (i) is (1.3). Relation (ii) follows from differentiating (i) and using (1.2), and (iii) follows from (ii) similarly.

Proposition 2.2. *Let*

$$A_m(\lambda, t) = \lambda^m S_\lambda((u - t)^m, t) = \lambda^m \int_{-\infty}^{\infty} W(\lambda, t, u)\,(u - t)^m\, du. \tag{2.1}$$

*Then*

$$A_{m+1}(\lambda, t) = \lambda m p(t)\, A_{m-1}(\lambda, t) + p(t)\frac{d}{dt} A_m(\lambda, t), \tag{2.2}$$

*and*

(i) *$A_m(\cdot, t)$ is a polynomial in $\lambda$.*

(ii) *The degree of $A_m(\cdot, t)$ is $[m/2]$.*

(iii) *The coefficients of the leading term $\lambda^m$ in $A_{2m}(\lambda, t)$ and $A_{2m+1}(\lambda, t)$ are $c_{1m}p^m(t)$ and $c_{2m}p^m(t)\,p'(t)$ respectively, where $c_{1m}$ and $c_{2m}$ are constants.* (2.3)

*Proof.* Relation (2.2) follows from differentiating (2.1) and using (1.2). The rest of the statement follows from (2.2), Proposition 2.1 and induction.

Proposition 2.3. *The sequence $\{S_\lambda((u - t)^m, t)\}_{m=0}^{\infty}$ has the generating function*

$$\sum_{m=0}^{\infty} S_\lambda((u - t)^m, t)\frac{x^m}{m!} = \exp\left\{-xt + \lambda \int_t^{g(q(t)+x/\lambda)} \frac{\theta}{p(\theta)}\, d\theta\right\}, \tag{2.4}$$

*where*

$$q(t) = \int_c^t \frac{dv}{p(v)}, \qquad c \in (A, B), \tag{2.5}$$

*and*

$$g(q(t)) = q(g(t)) = t. \tag{2.6}$$

*Proof.* First we rewrite (2.2) in terms of $S_\lambda((u-t)^m, t)$, that is

$$\frac{\lambda}{p(t)} S_\lambda((u-t)^{m+1}, t) = mS_\lambda((u-t)^{m-1}, t) + \frac{d}{dt} S_\lambda((u-t)^m, t), \tag{2.7}$$

then solve (2.7) by using the exponential generating function

$$G(t, z) = \sum_{m=0}^{\infty} \frac{z^m}{m!} S_\lambda((u-t)^m, t).$$

Multiplying (2.7) by $z^m/m!$ and summing over $m = 0, 1, \ldots$, we get

$$\frac{\lambda}{p(t)} \frac{\partial G(t, z)}{\partial z} - \frac{\partial G(t, z)}{\partial t} = zG(t, z). \tag{2.8}$$

We now solve (2.8) by the method of characteristics, so we solve the system

$$\frac{p(t)}{\lambda} dz = \frac{dt}{-1} = \frac{dG}{zG}.$$

We have: $z = -\lambda q(t) + C_1$ and $dG/dt = -zG = (\lambda q(t) - C_1)\, G$, hence

$$G(t, z) = C_2 \exp\left\{-C_1 t + \lambda \int_c^t q(\theta)\, d\theta\right\} = C_2 \exp\left\{-zt - \lambda \int_c^t \frac{v\, dv}{p(v)}\right\},$$

because

$$\int_c^t q(\theta)\, d\theta = \int_c^t \int_c^\theta \frac{dv\, d\theta}{p(v)} = \int_c^t \int_v^t \frac{d\theta\, dv}{p(v)} = tq(t) - \int_c^t \frac{v\, dv}{p(v)}.$$

Therefore the general solution of (2.8) is

$$G(t, z) = \exp\left\{-zt - \lambda \int_c^t \frac{v\, dv}{p(v)}\right\} \phi(z + \lambda q(t)),$$

$\phi$ is arbitrary. The initial condition $G(t, 0) = 1$ gives

$$\phi(\lambda q(t)) = \exp\left\{\lambda \int_c^t \frac{v\, dv}{p(v)}\right\},$$

that is

$$\phi(\lambda t) = \exp\left\{\lambda \int_c^{g(t)} \frac{v\, dv}{p(v)}\right\},$$

and we see that $G(t, z)$ is nothing but the right hand side of (2.4).

So far the above analysis is formal but it leads to a right guess that requires a rigourous proof. Recall that (1.2) holds in $(A, B)$ and $p(t)$ is positive there.

The analyticity and positivity of $p(t)$ on $(A, B)$ imply that $q(t)$ of (2.5) is analytic and strictly increasing on $(A, B)$. Furthermor $q'(t) = 1/p(t) \neq 0$, shows that the inverse of $q(t)$, namely $g(t)$, is analytic on the range of $q(t)$. The range of $q(t)$, that is the domain of $g(t)$, is a nonempty open interval, say $(C, D)$, since $q(t)$ is a continuous strictly increasing function on $(A, B)$. Now for every $t$, there will exist $\delta(t)$ such that $q(t) + (x/\lambda) \in (C, D)$ for $|x/\lambda| < \epsilon(t)$. Clearly $g(q(t) + (x/\lambda))$ is an analytic function of $x$ in the interval $|x| < \lambda\delta(t)$, and the right hand side of (2.4) will also be analytic in $x$ there. Via direct substitution, one sees that the right hand side of (2.4) also satisfies the differential equation (2.8). Set

$$\exp\left\{-xt + \lambda \int_t^{g(q(t)+x/\lambda)} \frac{\theta\, d\theta}{p(\theta)}\right\} = \sum_{n=0}^{\infty} a_n(\lambda, t)\, x^n, \qquad |x| < \lambda \in (t), \tag{2.4$'$}$$

and substitute this power series expansion in (2.8) and equate the coefficients of like powers to get

$$\frac{\lambda}{p(t)}\, a_{m+1}(\lambda, t) = m a_{m-1}(\lambda, t) + \frac{d}{dt}\, a_m(\lambda, t). \tag{2.7$'$}$$

From (2.4) it follows that $a_0(\lambda, t) = S_\lambda((u - t)^0, t) = 1$ and $a_1(\lambda, t) = S_\lambda(u - t, t) = 0$. This shows that both $a_m(\lambda, t)$ and $S_\lambda((u - t)^m, t)$ satisfy the same recurrence relation, (2.7) or (2.7)$'$, and the initial conditions $a_j(\lambda, t) = \delta_{j,0}$, $j = 0, 1$. On the other hand it is obvious that (2.7) generates a unique sequence with prescribed $a_0$ and $a_1$. Thus the $a$'s and the $S$'s are identical for all $t$. This completes the proof.

*Remark* 2.4. The generating function (2.4) provides an alternate proof of (i), (ii) and (iii) in Proposition 2.2.

PROPOSITION 2.5. *Let $N > 0$ and $A < a < b < B$. Then*

$$\| S_\lambda(e^{N|u|}, t)\|_{C[a,b]} < \infty, \tag{2.9}$$

*for large* $\lambda$.

*Proof.*

$$S_\lambda(e^{Nu}, t) = e^{Nt} S_\lambda(e^{N(u-t)}, t) = e^{Nt} \sum_{m=0}^{\infty} \frac{N^m}{m!} S_\lambda((u - t)^m, t)$$

$$= \exp\left\{\lambda \int_t^{g(q(t)+N/\lambda)} \frac{\theta}{p(\theta)}\, d\theta\right\}$$

for sufficiently large $\lambda$, hence

$$\| S_\lambda(e^{Nu}, t)\|_{C[a,b]} < \infty. \tag{2.10}$$

Similarly

$$\| S_\lambda(e^{-Nu}, t)\|_{C[a,b]} < \infty. \tag{2.11}$$

Now (2.10) and (2.11) imply (2.9) since $e^{N|u|} < e^{Nu} + e^{-Nu}$ and $S_\lambda(\cdot, t)$ is a positive linear operator.

*Remark* 2.6. The term "growth-test function" is introduced in [7] as a positive function $\psi$ such that $S_\lambda(\psi^2, t) < \infty$. It is shown there that $\psi$ can be chosen at least as any polynomial.

The result of the last proposition shows that $\psi(u)$ can be chosen as any $e^{N|u|}$, $N > 0$. This improvement is useful in applications. Moreover, this result is also best possible. For example, if we take $S_\lambda = P_\lambda$ to be the Post–Widder operator (cf. (3.9) later),

$$P_\lambda(f, t) = \frac{(\lambda/t)^\lambda}{\Gamma(\lambda)} \int_0^\infty e^{-\lambda u/t} u^{\lambda-1} f(u)\, du,$$

then it is clear that

$$P_\lambda(e^{u^{1+\delta}}, t) = \infty \qquad \text{for any } \delta > 0.$$

COROLLARY 2.7. *Let $\eta$, $k$ and $N$ be three positive numbers. Let $[a, b] \subset (A, B)$. Then*

(i) $\lim_{\lambda\to+\infty} S_\lambda(e^{Nu}, t) = e^{Nt}$ *in* $C[a, b]$

(ii) $\int_{|u-t|\geqslant\eta} W(\lambda, t, u)\, e^{Nu}\, du = O(\lambda^{-k})$ *uniformly on* $[a, b]$, *as* $\lambda \to +\infty$.

DEFINITION 2.8. The space $(C_N, \|\cdot\|_{C_N})$ is defined as

$$C_N = \{f \in C(A, B);\ |f(t)| \leqslant Me^{N|t|} \text{ for some } M \geqslant 0\}$$

and

$$\|f\|_{C_N} = \sup\{e^{-N|t|}\, |f(t)|;\ t \in (A, B)\}.$$

PROPOSITION 2.9 (Voronovskaja-type relation). *Let $f \in C_N$. If for some $\xi \in (A, B)$, $f''(\xi)$ exists, then*

$$\lim_{\lambda\to\infty} \lambda(S_\lambda(f, \xi) - f(\xi)) = \tfrac{1}{2}p(\xi) f''(\xi). \tag{2.12}$$

*If $f \in C^2[a, b]$, then the convergence in* (2.12) *is uniform in any interior interval $[a_1, b_1] \subset (a, b)$.*

*Proof.* First we use Taylor's formula to derive

$$\begin{aligned} S_\lambda(f, \xi) - f(\xi) &= S_\lambda(f'(\xi)(u - t) + \tfrac{1}{2} f''(\xi)(u - \xi)^2 + \epsilon(u, \xi)(u - t)^2, \xi) \\ &= \frac{p(\xi)}{2\lambda} f''(\xi) + \int_A^B W(\lambda, t, u)\, \epsilon(u, \xi)\, (u - t)^2\, du \\ &= \frac{p(\xi)}{2\lambda} f''(\xi) + I. \end{aligned}$$

The second term is dominated by $\epsilon + O(1/\lambda)$ by Corollary 2.7.

The details of the above proof can be found, for example, in [5, p. 22].

## 3. Exponential Operators

The first result in this section is that the kernel $W(\lambda, t, u)$ of an exponential operator is characterized by (1.2) and (1.3). Once given the differential equation (1.2) we take the domain of $t$ to be any component (that is a maximal connected subset) of the set $\{t: p(t) > 0, p(t) \text{ analytic}\}$, and the domain of $u$ to be $(-\infty, \infty)$.

Proposition 3.1. *The partial differential equation* (1.2) *and the normalization condition* (1.3) *define at most one kernel* $W(\lambda, t, u)$.

*Proof.* Let $W(\lambda, t, u)$ satisfy (1.2) and let

$$\xi(\lambda, t, u) = \exp\left\{-\lambda \int_c^t \frac{u-\theta}{p(\theta)}\, d\theta\right\} W(\lambda, t, u).$$

The above $\xi(\lambda, t, u)$ may be a generalized function. Using (1.2) we get

$$\frac{\partial \xi(\lambda, t, u)}{\partial t} = 0,$$

hence $\xi(\lambda, t, u)$ is independent of $t$, that is

$$W(\lambda, t, u) = \exp\left\{\lambda \int_c^t \frac{u-\theta}{p(\theta)}\, d\theta\right\} C(\lambda, u). \tag{3.1}$$

The normalization (1.3) yields

$$\exp\left(\lambda \int_c^t \frac{\theta\, d\theta}{p(\theta)}\right) = \int_{-\infty}^{\infty} C(\lambda, u) \exp(\lambda u q(t))\, du, \tag{3.2}$$

which implies that

$$\exp\left(\lambda \int_c^{g(x)} \frac{\theta\, d\theta}{p(\theta)}\right) = \int_{-\infty}^{\infty} e^{\lambda u x} C(\lambda, u)\, du, \qquad x \in \text{Range of } q(t), \tag{3.3}$$

where $q(t)$ and $g(t)$ are as in (2.5) and (2.6) respectively. Now one finds $C(\lambda, u)$, which may be a generalized function, by inverting the two sided (or bilateral) Laplace transform (3.3). There is at most one $C(\lambda, u)$ satisfying (3.3), see for example Zemanian [13, p. 69]. Note that (3.3) is satisfied if and only if

$$\exp\left(\lambda \int_c^{g(z)} \frac{\theta\, d\theta}{p(\theta)}\right) = \int_{-\infty}^{\infty} e^{\lambda u z} C(\lambda, u)\, du, \qquad \operatorname{Re} z \in \text{Range of } q(t). \tag{3.4}$$

Corollary 3.2. *Any solution of the differential equation* (1.2) *is of the form* (3.1).

We proceed to characterize all the approximation processes with $p(t)$ a quadratic polynomial. When $p(t)$ is a positive constant we may assume $p(t) \equiv 1$ since the constant can be absorbed in $\lambda$. We take $c = 0$, hence $q(t) = g(t) = t$ and (3.3) becomes

$$\exp(\lambda x^2/2) = \int_{-\infty}^{\infty} e^{\lambda u x} C(\lambda, u)\, du, \qquad x \in (-\infty, \infty).$$

Therefore

$$C(\lambda, u) = \sqrt{\frac{\lambda}{2\pi}} \exp(-\lambda u^2/2), \qquad u \in (-\infty, \infty),$$

and the approximation operator $S_\lambda(f, t)$, when $p(t) = 1$, is the Gauss–Weierstrass operator

$$W_\lambda(f, t) = \sqrt{\frac{\lambda}{2\pi}} \int_{-\infty}^{\infty} \exp\{-\lambda(u - t)^2/2\} f(u)\, du, \qquad t \in (-\infty, \infty). \tag{3.5}$$

Consider next the case of linear $p(t)$. There is no loss of generality in assuming $p(t) = \pm t$ since otherwise we can make a linear change of both the variables $u$ and $t$. If $p(t) = t$ we take $c = 1$, so that $q(t) = \ln t$, $g(t) = e^t$ and (3.2) becomes

$$\exp(\lambda e^x - \lambda) = \int_{-\infty}^{\infty} e^{\lambda u x} c(\lambda, u)\, du, \qquad x > 0.$$

Therefore

$$C(\lambda, u) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \delta(k - \lambda u),$$

and we get, by (3.1), the Szász operator

$$S_\lambda(f, t) = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} f\left(\frac{k}{\lambda}\right), \qquad t \in (0, \infty). \tag{3.6}$$

The case $p(t) = -t$, generates an equivalent operator with $t \in (-\infty, 0)$.

When $p(t)$ is a quadratic with two real distinct roots and $p(\infty) = -\infty$, we can assume, via a change of variables if necessary, that $p(t) = t(1 - t)$ with $t \in (0, 1)$. We take $c = \frac{1}{2}$, so that $q(t) = \ln\{t/(1 - t)\}$ and $g(t) = e^t/(1 + e^t)$. Formula (3.4) becomes

$$\left(\frac{1 + e^z}{2}\right)^\lambda = \int_{-\infty}^{\infty} e^{\lambda u z} C(\lambda, u)\, du,$$

for all $z$ in the complex plane. In order for $C(\lambda, u)$ to exist, it is necessary that $((1 + e^z)/2)^\lambda$ to be analytic for all complex $z$, see Zemanian [13, p. 70]. This is possible only if $\lambda$ is an integer, otherwise the aforementioned function will have branch points, for example, at $z = \pi i$. This and (3.1) prove that the approximation operator is the Bernstein polynomial operator [5]

$$B_n(f, t) = \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} f\left(\frac{k}{n}\right), \qquad n = 1, 2, \ldots, \quad t \in [0, 1]. \tag{3.7}$$

We now turn to the case $p(t)$ a quadratic with two distinct real roots but $p(t) = +\infty$. We can assume $p(t) = t(1 + t)$. Take $c = 1$, $t \in (0, \infty)$ and $q(t) = \ln\{2t/(1 + t)\}$, $g(t) = e^t/(2 - e^t)$. The inversion of

$$(2 - e^x)^{-\lambda} = \int_{-\infty}^{\infty} e^{\lambda ux} C(\lambda, u)\, du, \qquad x \in (-\infty, \ln 2),$$

is

$$C(\lambda, u) = 2^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda)_k}{k!} 2^{-k} \delta(k - \lambda u),$$

where

$$\begin{aligned} (\lambda)_k &= 1 && \text{if} \quad k = 0, \\ &= \lambda(\lambda + 1) \cdots (\lambda + k - 1) && \text{if} \quad k > 0. \end{aligned}$$

Therefore this case leads to the approximation operator

$$L_\lambda(f, t) = (1 + t)^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda)_k}{k!} \left(\frac{t}{1 + t}\right)^k f\left(\frac{k}{\lambda}\right), \qquad t \in (0, \infty), \tag{3.8}$$

introduced by Baskakov [1]. If we choose the component $(-\infty, -1)$ instead of $(0, \infty)$ we get an equivalent operator.

When $p(t)$ has a double real root, we can assume that $p(t) = t^2$. Choose $c = 1$. In the present case $q(t) = (t - 1)/t$, $g(t) = 1/(1 - t)$, and $C(\lambda, u)$ satisfies

$$(1 + z)^{-\lambda} = \int_{-\infty}^{\infty} e^{z\lambda u} C(\lambda, u)\, du, \qquad \operatorname{Re} z \in (-\infty, 1).$$

This proves that

$$\begin{aligned} C(\lambda, u) &= \frac{\lambda e^{-u\lambda}}{\Gamma(\lambda)} (u\lambda)^{\lambda - 1}, && u > 0, \\ &= 0, && u \leqslant 0, \end{aligned}$$

and that the corresponding approximation operator is the Post–Widder operator [12. p. 288]

$$P_\lambda(f, t) = \frac{(\lambda/t)^\lambda}{\Gamma(\lambda)} \int_0^{\infty} e^{-(\lambda u/t)} u^{\lambda - 1} f(u)\, du, \qquad t \in (0, \infty). \tag{3.9}$$

This is in fact equivalent to the gamma operator [6].

The only case left is when $p(t)$ has two complex zeros. Now we take the polynomial $p(t)$ to be $t^2 + 1$. The functions $q(t)$ and $g(t)$ are arctan $t$ and tan $t$ respectively, by choosing $c = 0$. The function $C(\lambda, u)$ satisfies

$$(\sec z)^\lambda = \int_{-\infty}^{\infty} e^{\lambda u z} C(\lambda, u)\, du, \qquad \operatorname{Re} z \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

In order to invert the above transform we need the following lemma.

Lemma 3.3. *We have*

$$(\sec z)^\lambda = \frac{2^{\lambda-2}}{\pi \Gamma(\lambda)} \int_{-\infty}^{\infty} e^{zt} \left| \Gamma\left(\frac{\lambda + it}{2}\right) \right|^2 dt, \qquad \lambda > 0, \quad \operatorname{Re} z \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

*Proof.* The above identity holds for real $z$, see Oberhettinger [10, Section 9, p. 46]. The function $(\sec z)^\lambda$ is analytic in the strips $\operatorname{Re} z \in [a, b]$, $[a, b] \in (-\pi/2, \pi/2)$. On the other hand the right hand side of the above identity is also defined and analytic in the same strips, since

$$| \Gamma(x + iy)|^2 \sim (2\pi)^{-1} \, | y |^{2x-1} \, e^{-\pi |y|}, \qquad \text{as} \quad | y | \to \infty,$$

see Erdelyi *et al.* [4, (16), p. 47]. The result now follow by the identity theorem for analytic functions.

As an immediate consequence of Lemma 3.3 we get

$$C(\lambda, u) = \frac{2^{\lambda-2}\lambda}{\pi \Gamma(\lambda)} \left| \Gamma\left(\frac{\lambda}{2} + \frac{i\lambda u}{2}\right) \right|^2.$$

The corresponding approximation operator is

$$T_\lambda(f, t) = \frac{2^{\lambda-2}\lambda}{\pi \Gamma(\lambda)} (1 + t^2)^{-\lambda/2} \int_{-\infty}^{\infty} e^{\lambda u \arctan t} \left| \Gamma\left(\frac{\lambda}{2} + \frac{i\lambda u}{2}\right) \right|^2 f(u)\, du, \qquad t \in (-\infty, \infty). \tag{3.10}$$

Thes we proved

Theorem 3.3. *The approximation operator defined by* (1.1) *whose kernels* $W(\lambda, t, u)$ *satisfy* (1.2) *and* (1.3) *with quadratic* $p(t)$ *are the Gauss–Weierstrass operator* ($p(t) = 1$), *the Száas operator* ($p(t) = t$), *the Bernstein operator* (*polynomials*) ($p(t) = t(1 - t)$), *the Baskakov operator* ($p(t) = t(1 + t)$), *the Post–Widder operator* ($p(t) = t^2$) *and the new operator* $T_\lambda(f, t)$ ($p(t) = 1 + t^2$), *as defined by* (3.5)–(3.10) *respectively.*

Meyer–König and Zeller [8] introduced the operators

$$M_\lambda(f, t) = (1 - t)^\lambda \sum_{k=0}^{\infty} \binom{k + \lambda - 1}{k} t^k f\left(\frac{k}{k + \lambda}\right), \qquad \lambda > 1.$$

Clearly the corresponding kernel is

$$W(\lambda, t, u) = (1-t)^{\lambda} t^{\lambda u/(1-u)} \sum_{k=0}^{\infty} \binom{k+\lambda-1}{k} \delta\left(u - \frac{k}{k+\lambda}\right).$$

Therefore

$$\frac{\partial}{\partial t} W(\lambda, t, u) = \frac{\lambda}{t}\left[\frac{u}{1-u} - \frac{t}{1-t}\right] W(\lambda, t, u).$$

We change the variables $t$ and $u$ to $x = t/(1-t)$ and $v = u/(1-u)$. Let $W_1(\lambda, x, v) = W(\lambda, t, u)$. It is clear that

$$\frac{\partial W_1(\lambda, x, v)}{\partial x} = \frac{\lambda(v-x)}{x(x+1)} W_1(\lambda, x, v).$$

In other words $W_1(\lambda, x, v)$ is the kernel of the Baskakov operator (3.8). More explicitely, we have

$$M_\lambda(f, t) = L_\lambda\left(f_1, \frac{t}{1-t}\right),$$

where $f_1(u) = f(u/(1+u))$.

We now introduce two new approximation processes corresponding to $p(t) = t^3$ and $t(1+t)^2$. We made no attempt to characterize all approximation operators generated by all possible cubic $p(t)$. Consider the case $p(t) = t^3$, $c = 1$. Then $q(t) = \frac{1}{2}(t^2 - 1)\, t^{-2}$, $g(t) = 1/(1-2t)^{1/2}$. In this case (3.4) becomes

$$\exp\{\lambda(1-(1-2z)^{1/2})\} = \int_{-\infty}^{\infty} e^{\lambda u z} C(\lambda, u)\, du, \qquad \operatorname{Re} z \in (-\infty, \tfrac{1}{2}),$$

yielding (see [3, p. 245, formula 1]),

$$\begin{aligned} C(\lambda, u) &= 0 && \text{for } u \in (-\infty, 0),\\ &= \left(\frac{\lambda}{2\pi}\right)^{1/2} u^{-3/2} \exp\left\{\lambda - \frac{1}{2}\lambda u - \frac{\lambda}{2u}\right\} && \text{for } u \in (0, \infty), \end{aligned}$$

and we obtain the new approximation operator

$$Q_\lambda(f, t) = \left(\frac{\lambda}{2\pi}\right)^{1/2} e^{\lambda/t} \int_0^{\infty} u^{-3/2} \exp\left(-\frac{\lambda u}{2t^2} - \frac{\lambda}{2u}\right) f(u)\, du, \qquad t \in (0, \infty). \tag{3.11}$$

The second example is $p(t) = t(1+t)^2$. We choose $c = 1$ and $q(t) = \ln(2t/(1+t)) + \frac{1}{2}(1-t)/(1+t)$. The generalized function $C(\lambda, u)$ satisfies (3.2), that is

$$\exp\left\{\frac{\lambda}{2}(t-1)/(t+1)\right\} = \int_{-\infty}^{\infty} \left(\frac{2t}{1+t}\right)^{\lambda u} \exp\left\{\frac{1}{2}\lambda u(1-t)/(1+t)\right\} C(\lambda, u)\, du,$$
$$t \in (0, \infty),$$

so we have to invert

$$e^{\lambda x} = \int_{-\infty}^{\infty} x^{\lambda u} e^{-x\lambda u}\, 2^{\lambda u} e^{\lambda(1+u)/2} C(\lambda, u)\, du, \qquad x \in (0, 1), \tag{3.12}$$

where $x = t/(1 + t)$. Recall that (see [9, p. 348]),

$$e^{\lambda z} = \sum_{k=0}^{\infty} \frac{\lambda(\lambda + k)^{k-1}}{k!} (ze^{-z})^k, \qquad \lambda \neq 0. \tag{3.13}$$

The above identity is an easy consequence of Lagrange's expansion theorem. A comparison between (3.12) and (3.13) yields that

$$C(\lambda, u) = 2^{-\lambda u} e^{-\lambda(1+u)/2} \sum_{k=0}^{\infty} \frac{\lambda(\lambda + k)^{k-1}}{k!} \delta(k - \lambda u).$$

The explicit form of the approximation operator is, for $\lambda > 0$,

$$R_\lambda(f, t) = e^{-\lambda t/(1+t)} \sum_{k=0}^{\infty} \frac{\lambda(\lambda + k)^{k-1}}{k!} \left(\frac{t}{1 + t}\right)^k e^{-kt/(1+t)} f\left(\frac{k}{\lambda}\right), \qquad t \in (0, \infty). \tag{3.14}$$

Note that the change of variables $x = t/(1 + t)$ leads to the equivalent approximation operator

$$R_\lambda^*(f, x) = e^{-\lambda x} \sum_{k=0}^{\infty} \frac{\lambda(\lambda + k)^{k-1}}{k!} (xe^{-x})^k f\left(\frac{k}{\lambda + k}\right), \qquad x \in (0, 1).$$

We finally come to our last example $p(t) = 2t^{3/2}$. We take $c = 1$ and call the new operator $\mathscr{T}_\lambda(f, t)$. Easy manipulations lead to

$$\mathscr{T}_\lambda(f, t) = e^{-\lambda\sqrt{t}} \int_{-\infty}^{\infty} e^{-\lambda u/\sqrt{t}} D(\lambda, u) f(u)\, du,$$

where $D(\lambda, u)$ will be determined from $\mathscr{T}_\lambda(1, t) \equiv 1$, that is

$$e^{\lambda\sqrt{t}} = \int_{-\infty}^{\infty} e^{-\lambda u/\sqrt{t}} D(\lambda, u)\, du. \tag{3.15}$$

From (3.15) we get, via [3, (31), p. 244],

$$\begin{aligned} D(\lambda, u) - \delta(u) &= \lambda u^{-1/2} I_1(2\lambda \sqrt{u}), \qquad && u \geqslant 0, \\ &= 0, && u < 0. \end{aligned}$$

So our new operator is given explicitly by

$$\mathscr{T}_\lambda(f, t) = e^{-\lambda\sqrt{t}} \left\{ f(0) + \lambda \int_0^{\infty} e^{-\lambda u/\sqrt{t}} u^{-1/2} I_1(2\lambda \sqrt{u}) f(u)\, du \right\} \tag{3.16}$$

where $I_1(x)$ is a modified Bessel function of the first kind.

## 4. Characterizing Functions by the Degree of Approximation

In this section, we characterize functions such that $\| S_\lambda(f, t) - f(t)\| = O(\lambda^{-\alpha/2})$, where $0 < \alpha < 2$. Since the proofs are similar to [7], we only sketch them. For more details, we refer the reader to [7].

Definition 4.1. Let $A < a < b < B$. The second modulus of smoothness of $f$ is defined as

$$\omega_2(f; h, a, b) = \sup\{| \Delta_\delta^2 f(t)|; t, t + 2\delta \in [a, b], | \delta | \leqslant h\}, \tag{4.1}$$

where

$$\Delta_\delta^2 f(t) = f(t) - 2f(t + \delta) + f(t + 2\delta). \tag{4.2}$$

Theorem 4.2 (Direct Theorem). *Let $A < a < a_1 < b_1 < b < B$, and $f \in C_N$. Then for every $m > 0$, there is a constant $K_m$, such that*

$$\| S_\lambda(f, t) - f(t)\|_{C[a_1, b_1]} \leqslant K_m[\omega_2(f; \lambda^{-1/2}, a, b) + \lambda^{-m} \| f \|_{C_N}]. \tag{4.3}$$

*Proof.* Let $\delta > 0$, define

$$g_\delta(x) = \frac{1}{2\delta^2} \int_{-\delta/2}^{\delta/2} \int_{-\delta/2}^{\delta/2} [f(x + u + v) + f(x - u - v)]\, du\, dv, \tag{4.4}$$

where $\delta \leqslant \frac{1}{2} \min\{a_1 - a, b - b_1\}$.

First, by linearity of $S_\lambda(\cdot, t)$, we find

$$\begin{aligned} &\| S_\lambda(f, t) - f(t)\|_{C[a_1, b_1]} \\ &\quad \leqslant \| S_\lambda(f - g_\delta, t)\|_{C[a_1, b_1]} + \| S_\lambda(g_\delta, t) - g_\delta(t)\|_{C[a_1, b_1]} + \| g_\delta(t) - f(t)\|_{C[a_1, b_1]} \\ &\quad = J_1 + J_2 + J_3 . \end{aligned}$$

Observe that, for $x \in [a_1, b_1]$,

$$| f(x) - g_\delta(x)| \leqslant \tfrac{1}{2}\omega_2(f; \delta, a, b).$$

Hence

$$J_3 \leqslant \tfrac{1}{2}\omega_2(f; \delta, a, b). \tag{4.5}$$

The estimation for $J_1$ follows from (4.5) and Corollary 2.7:

$$\begin{aligned} J_1 &\leqslant \left\| \left[\int_{|u-t|<\eta} + \int_{|u-t|\geqslant\eta}\right] w(\lambda, t, u)\, | f(u) - g_\delta(u)|\, du \right\|_{C[a_1, b_1]} \\ &\leqslant K_m(\omega_2(f; \delta, a_1, b_1) + \lambda^{-m} \| f \|_{C_N}), \end{aligned} \tag{4.6}$$

where $\eta = \frac{1}{2} \min\{a_1 - a, b - b_1\}$.

The estimation for $J_2$ is done similarly because

$$g_\delta''(x) = \delta^{-2}\bar{\Delta}_\delta^2 f(x),$$

where $\bar{\Delta}_\delta^2$ is the second symmetric difference

$$\bar{\Delta}_\delta^2 f(x) = f(x-\delta) - 2f(x) + f(x+\delta),$$

and

$$\| g_\delta'' \|_{C[a_1-\eta, b_1+\eta]} \leqslant \delta^{-2}\omega_2(f; \delta, a, b),$$

hence

$$J_2 \leqslant \left\| \left[ \int_{|u-t|<\eta} + \int_{|u-t|\geqslant\eta} \right] W(\lambda, t, u) g_\delta''(\beta)(u-t)^2\, du \right\|_{C[a_1, b_1]}.$$

Using the same argument as in estimating $J_1$, we obtain

$$J_2 \leqslant K_m[\delta^{-2}\lambda^{-1}\omega_2(f; \delta, a, b) + \lambda^{-m} \| f \|_{C_N}]. \tag{4.7}$$

The theorem then follows from (4.5), (4.6) and (4.7) by setting $\delta = \lambda^{-1/2}$.

In particular, this theorem shows that, if $\omega_2(f, \delta, a, b) = O(\delta^\alpha)$, then $\| S_\lambda(f, t) - f(t) \|_{C[a_1, b_1]} = O(\lambda^{-\alpha/2})$.

Note that the term $\lambda^{-m} \| f \|_{C_N}$ in (4.3) cannot be omitted. For example, let $S_n = B_n$, the Bernstein polynomial, $a = \frac{1}{3}$, $b = \frac{2}{3}$, and $f(x) = (\frac{1}{3} - x)_+$. Clearly, $B_n(f, t) \neq 0$, but $\omega_2(f, \lambda^{-1/2}, a, b) = 0$.

The converse of this theorem, the inverse theorem, is proved next.

We first define the Zygmund class of functions.

Definition 4.3. Let $0 < \alpha < 2$. The Zygmund class Lip* $\alpha$ is defined as

$$\mathrm{Lip}^*(\alpha; a, b) = \{f;\ \omega_2(f; h, a, b) \leqslant Mh^\alpha\}.$$

Theorem 4.4. *Let* $A < a_1 < a_2 < b_2 < b_1 < B$, $0 < \alpha < 2$ *and* $f \in C_N$. *Then in the following statements* (i) *implies* (2).

(i) $\| S_\lambda(f, t) - f(t) \|_{C[a_1, b_1]} = O(\lambda^{-\alpha/2})$;

(ii) $f \in \mathrm{Lip}^*(\alpha; a_2, b_2)$.

The proof is divided into two parts. We first prove the special case when $f$ is of compact support strictly contained inside some open interval $(a, b)$, and then pass to the general case.

(I) Let $A < a < a' < a'' < b'' < b' < b < B$. Assume $f \in C_0$ with support in $[a'', b'']$. Denote

$$K(\xi, f) = \inf\{\| f - g \| + \xi \| g'' \|;\ g \in \mathscr{G}\}, \qquad 0 < \xi \leqslant 1, \tag{4.8}$$

where $\mathscr{G} = \{g \in C_0^2,\ \operatorname{supp} g \subset [a', b']\}$.

LEMMA 4.5. *If* $\| S_\lambda(f, t) - f(t)\|_{C[a,b]} \leqslant M_1\lambda^{-\alpha/2}$, *then*

$$K(\xi, f) \leqslant M_2[\lambda^{-\alpha/2} + \lambda\xi K(\lambda^{-1}, f)]. \tag{4.9}$$

*Proof.* It is enough to show that there exists an $M$, such that, for each $g \in \mathscr{G}$,

$$\| S''_\lambda(f, t)\|_{C[a,b]} \leqslant M\lambda[\| f - g \| + \lambda^{-1} \| g'' \|]. \tag{4.10}$$

In fact

$$\begin{aligned}\| S''_\lambda(f, t)\|_{C[a,b]} &\leqslant \| S''_\lambda(f - g, t)\|_{C[a,b]} + \| S''_\lambda(g, t)\|_{C[a,b]}\\ &= I_1 + I_2 .\end{aligned}$$

Clearly, $I_1 \leqslant M_1\lambda \| f - g \|$, for $\operatorname{supp}(f - g) \subset [a', b']$, and

$$\frac{\partial^2}{\partial t^2} W(\lambda, t, u) = \left[\frac{\lambda^2}{p^2(t)} (u - t)^2 + \lambda \left(\frac{\partial}{\partial t} \frac{u - t}{p(t)}\right)\right] W(\lambda, t, u), \tag{4.11}$$

and the estimation follows from Proposition 2.2. On the other hand, observe $S''_\lambda(1, t) = S''_\lambda(u, t) = 0$, hence, using Taylor's formula on $g$,

$$S''_\lambda(g, t) = \int \left[\frac{\partial^2}{\partial t^2} W(\lambda, t, u)\right] g''(\xi)(u - t)^2\, du,$$

and by (4.12) and Proposition 2.2, $\| S''_\lambda(g, t)\|_{C[a,b]} \leqslant M \| g'' \|$.

LEMMA 4.6. *Under the same conditions of Lemma* 4.5, *we have*

$$K(\xi, f) \leqslant M\xi^{\alpha/2}. \tag{4.12}$$

*Consequently*, $f \in \operatorname{Lip}^*(\alpha; a, b)$.

The proof is standard and can be found in [2].

(II) Now we prove the general case. That is, we assume that $A < a_1 < a_2 < b_2 < b_1 < B$, $f \in C_N$ and

$$\| S_\lambda(f, t) - f(t)\|_{C[a_1,b_1]} = O(\lambda^{-\alpha/2}). \tag{4.13}$$

The proof is also divided into two parts.

*Case* 1. $0 < \alpha \leqslant 1$.

Let $a_1 < a' < a'' < a_2 < b_2 < b'' < b' < b_1$. Let $g \in C_0^\infty$ such that $\operatorname{supp} g \subset [a'', b'']$, $g(x) = 1$ on $[a_2, b_2]$.

LEMMA 4.7. *Suppose* (4.13) *holds for* $0 < \alpha \leqslant 1$, *and* $g$ *is defined as above, then*

$$\| S_\lambda(fg, t) - fg(t)\|_{C[a',b']} = O(\lambda^{-\alpha/2}). \tag{4.14}$$

*As consequences*, $fg \in \operatorname{Lip}^*(\alpha; a', b')$, *and* $f \in \operatorname{Lip}^*(\alpha; a_2, b_2)$.

*Proof.* For $t \in [a', b']$, we have

$$S_\lambda(fg, t) - f(t)\, g(t)$$
$$= g(t)\,[S_\lambda(f, t) - f(t)] + \int_{a_1}^{b_1} W(\lambda, t, u) f(u)\,[g(u) - g(t)]\, du + o\left(\frac{1}{\lambda}\right)$$
$$= I_1 + I_2 + o\left(\frac{1}{\lambda}\right).$$

The estimate for $I_1$ follows from (4.13):

$$\| I_1 \|_{[a',b']} \leqslant \| g \| \cdot \| S_\lambda(f, t) - f(t) \|_{C[a',b']} \leqslant M_1 \lambda^{-\alpha/2}.$$

The estimate for $I_2$ follows by the mean value theorem

$$\| I_2 \| \leqslant \int_{a_1}^{b_1} W(\lambda, t, u)\, | f(u)\, g'(\xi) |\, | u - t |\, du = O(\lambda^{-1/2}).$$

*Case* II. $1 < \alpha < 2$.

Let $a_1 < x < a'$ and $b' < y < b_1$, and $0 < \delta < \frac{1}{2}$. To prove the inverse theorem, it is enough to prove the following lemma.

LEMMA 4.8. *Suppose* (4.13) *holds for* $1 < \alpha < 2$, *and* $g$ *is defined above, then*

$$\| S_\lambda(fg, t) - fg(t) \|_{C[a',b']} = O(\lambda^{-\alpha/2}). \tag{4.15}$$

*Consequently,* $fg \in \mathrm{Lip}^*(\alpha;\, a', b')$ *and* $f \in \mathrm{Lip}^*(\alpha;\, a_2, b_2)$.

*Proof.* By step I, we may conclude $f \in \mathrm{Lip}(1 - \delta;\, x, y)$. Now for $t \in [a', b']$, we have

$$\begin{aligned} S_\lambda(fg, t) - f(t)\, g(t) &= g(t)\,[S_\lambda(f, t) - f(t)] + f(t)\,[S_\lambda(g, t) - g(t)] \\ &\quad + \int_x^y W(\lambda, t, u)\,[f(u) - f(t)]\,[g(u) - g(t)]\, du + o(\lambda^{-1}) \\ &= J_1 + J_2 + J_3 + o(\lambda^{-1}). \end{aligned}$$

Now $\| J_1 \|_{C[a',b']} = O(\lambda^{-\alpha/2})$ by (4.14), and $\| J_2 \|_{C[a',b']} = O(\lambda^{-1}) \leqslant O(\lambda^{-\alpha/2})$ since $g \in C^\infty$. Also from $| f(u) - f(t) | \leqslant M_1 \,| u - t |^{1-\delta}$ and $| g(u) - g(t) | = | g'(\xi)(u - t) | \leqslant M_2 \,| u - t |$, $\| J_3 \|_{C[a',b']} = O(\lambda^{-(2-\delta)/2}) \leqslant O(\lambda^{-1+\delta/2})$.

Since $\delta > 0$ can be chosen arbitrary small, we conclude that (4.15) holds for all $\alpha \in (1, 2)$.

## REFERENCES

1. V. A. BASKAKOV, An example of a sequence of linear positive operators in the space of continuous functions (Russian), *Dokl. Akad. Nauk* 113 (1957), 249–251.

2. H. BERENS AND G. G. LORENTZ, Inverse theorems for Bernstein polynomials, *Indiana Univ. J. Math.* **21** (1972), 693–708.
3. A. ERDÉLYI *et al.*, "Tables of Integrals Transforms," Vol. 1, McGraw-Hill, New York, 1954.
4. A. ERDÉLYI *et al.*, "Higher Transcendental Functions," Vol. 1, McGraw-Hill, New York, 1954.
5. G. G. LORENTZ, "Bernstein Polynomials," Univ. of Toronto Press, Toronto, 1953.
6. A. LUPAS AND M. MULLER, Approximationseigenschaften der Gammaoperatoren, *Math. Z.* **98** (1967), 208–226.
7. C. P. MAY, Saturation and Inverse theorems for combinations of a class of exponential type operators, *Canad. J. Math.* **28** (1976), 1224–1250.
8. W. MEYER-KÖNIG AND K. ZELLER, Bernsteinache Potenzreihen, *Studia Math.* **19** (1960), 89–94.
9. G. PÓLYA AND G. SZEGÖ, "Problems and Theorems in Analysis," Vol. 1 (English translation), Springer-Verlag, New York, 1972.
10. F. OBERHETTINGER, "Tabellen zur Fourier transformation," Springer-Verlag, Berlin, 1957.
11. O. SZASZ, Generalizations of S. Bernstein's polynomials to the infinite interval, *J. Res. Nat. Bur. Stand.* **45** (1950), 230–245.
12. D. V. WIDDER, "The Laplace Transform," Princeton Univ. Press, Princeton, 1941.
13. A. H. ZEMANIAN, "Generalized Integral Transformations," Interscience, New York, 1965.